Doporučená úloha č.19 (Zajíček)

$$Dokažte, že \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx = -\frac{\pi \ln 2}{2}$$

$$x = 2t \qquad \int_0^{\frac{\pi}{4}} \ln(\sin(2t))2\,dt$$

$$\sin(2t) = 2 * \sin t * \cos t = \ln(\sin(2t)) = \ln 2 + \ln(\sin t) + \ln(\cos t)$$

$$t = \frac{\pi}{2} - u \qquad \int_0^{\frac{\pi}{4}} \ln(\cos t)\,dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left(\cos\left(\frac{\pi}{2} - u\right)\right)\,du = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin u)\,du$$

$$\Rightarrow \quad 2 * \int_0^{\frac{\pi}{4}} \ln(\sin(2t))\,dt = 2 * \left[\int_0^{\frac{\pi}{4}} \ln 2\,dt + \int_0^{\frac{\pi}{4}} \ln(\sin(t))\,dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln(\sin(t))\,dt\right]$$

$$= 2 * \left[\frac{\pi}{4}\ln 2 + \int_0^{\frac{\pi}{2}} \ln(\sin(t))\,dt\right] = \Delta$$

$$I = \int_0^{\frac{\pi}{2}} \ln(\sin(t))\,dt$$

$$t = \frac{\pi}{2} - u \qquad \int_0^{\frac{\pi}{2}} \ln(\sin(t))\,dt = \int_{\frac{\pi}{2}}^{0} -\ln\left(\sin\left(\frac{\pi}{2} - u\right)\right)\,du = \int_0^{\frac{\pi}{2}} \ln(\cos(u))\,du = I$$

$$2I = \int_0^{\frac{\pi}{2}} \ln(\sin(t))\,dt + \int_0^{\frac{\pi}{2}} \ln(\cos(t))\,dt = \int_0^{\frac{\pi}{2}} \ln(\sin(t)\cos(t))\,dt$$

$$= \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin(2t)}{2}\right)\,dt = -\int_0^{\frac{\pi}{2}} \ln(2)\,dt + \int_0^{\frac{\pi}{2}} \ln(\sin(2t))\,dt = (\star)$$

$$u = 2t \qquad \int_0^{\frac{\pi}{2}} \ln(\sin(2t))\,dt = \int_0^{\pi} \ln(\sin(u)) * \frac{1}{2}\,du = \frac{1}{2} * \left[\int_0^{\frac{\pi}{2}} \ln(\sin(u))\,du + \int_{\frac{\pi}{2}}^{\pi} \ln(\sin(u))\,du\right]$$

$$v = \pi - u$$

$$= \frac{1}{2} * \left[I + \int_{\frac{\pi}{2}}^{0} -\ln(\sin(\pi - v))\,dv\right] = \frac{1}{2} * \left[I + \int_0^{\frac{\pi}{2}} \ln(\sin(v))\,dv\right] = \frac{1}{2} * [I + I] = I$$

$$(\star) = 2I = -\int_0^{\frac{\pi}{2}} \ln(2)\,dt + I \Rightarrow I = -\ln(2) * \frac{\pi}{2}$$

$$\Delta = 2 * \left[\ln(2) * \frac{\pi}{4} + (-\ln(2) * \frac{\pi}{2})\right] = \frac{\pi \ln(2)}{2} - \frac{2\pi \ln(2)}{2} = -\frac{\pi \ln(2)}{2}$$